Written Exam for the M.Sc. in Economics Autumn 2011-2012 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

Exam date: **11/1-2012**

3-hour open book exam.

Please note **all questions need to be answered**.

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Question A:

Question A.1: Figure A.1 shows a log-returns series y_t with $T = 1000$ observations together with the ACF for $|x_t|$. Furthermore, Table A.1 shows some output from estimation of a GARCH(1,1) model with these data.

From Figure A.1, would you expect a GARCH(1,1) to fit the data? Explain the output in Table A.1.

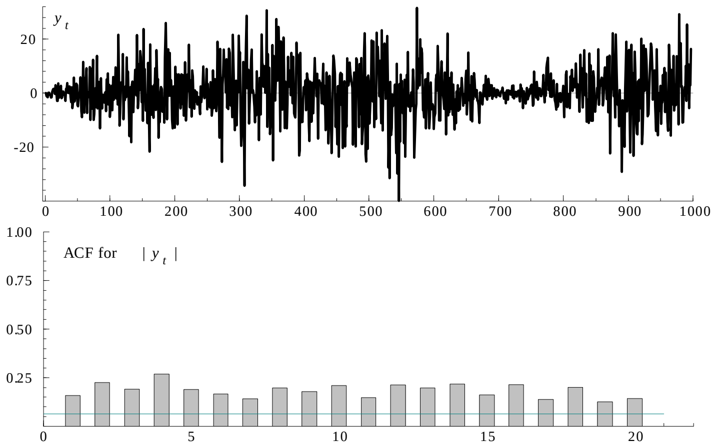

Figure A.1

Table A.1	
Parameter estimates in GARCH(1,1):	$\hat{\alpha} + \hat{\beta} = 1.03$
Standardized residuals: $\hat{z}_t = y_t / \hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	p-value: 0.00
LM ARCH test in $\hat{z}_t$:	p-value: 0.15

Question A.2: Theory suggests that the volatility of y_t is driven by the an observed factor x_t, and not by its own past as suggested by estimating the GARCH(1,1) model. We wish to investigate if indeed y_t's conditional volatility σ_t is driven by x_t and do so by considering the following model:

$$y_t = \sigma_t z_t \tag{1}$$
$$\sigma_t^2 = \omega + \alpha_x x_{t-1}^2 + \alpha_y y_{t-1}^2, \tag{2}$$

where z_t are iidN$(0, 1)$ and x_t is the observed process which drives the conditional variance of the returns y_t. The parameters to be estimated are given by $\theta = (\omega, \alpha_x, \alpha_y)$, with $\omega > 0$ and $\alpha_x, \alpha_y \geq 0$.

State the log-likelihood function $L_T(\theta)$ and show that $S_T(\theta) := \frac{1}{\sqrt{T}} \partial L_T(\theta) / \partial \alpha_x$, is given by (up to your choice of normalization),

$$S_T(\theta) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{1}{2} \left(\frac{y_t^2}{\sigma_t^2} - 1 \right) \left(\frac{x_{t-1}^2}{\sigma_t^2} \right).$$

Question A.3: Show that $S_T(\theta_0)$ is asymptotically Gaussian. When you establish this, explain which results you use, and how you use them. Also, state and discuss which conditions on the parameters θ_0 and on the exogenous process x_t you use.

Discuss briefly what is missing in order to conclude that $\sqrt{T}(\hat{\alpha}_x - \alpha_{x,0})$ is asymptotically Gaussian.

Question A.4:

Suppose that the factor x_t driving σ_t is given by an AR(1) process,

$$x_t = \rho x_{t-1} + \varepsilon_t, \tag{3}$$

where ε_t are iidN$(0, \sigma_\varepsilon^2)$ and the two innovation series ε_t and z_t are independent. The series x_t and its differences Δx_t are shown in Figure A.2. Comment on the output reported in Table A.2 using Figure A.2 and your previous results.

Table A.2	
Model:	Estimates/[std. error]
$\sigma_t^2 = \omega + \alpha_x x_t^2 + \alpha_y y_{t-1}^2$	$\hat{\alpha}_x = \underset{[0.021]}{0.4}$ $\hat{\alpha}_y = \underset{[0.029]}{0.03}$
Standardized residuals: $\hat{z}_t = y_t / \hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	p-value: 0.72
LM ARCH test in $\hat{z}_t$:	p-value: 0.85

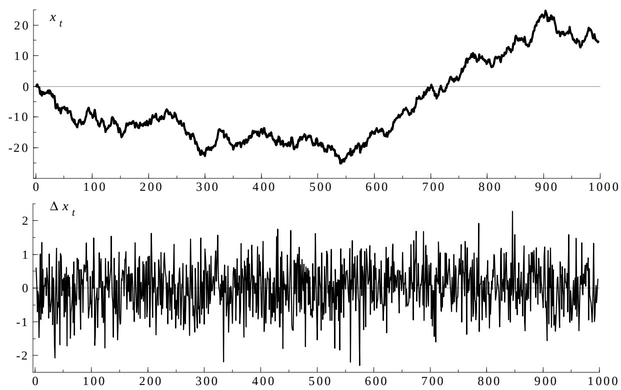

Figure A.2

Question B:

Question B.1: Consider the return series y_t in Figure B.1 with t=1,2,...,T=3000.

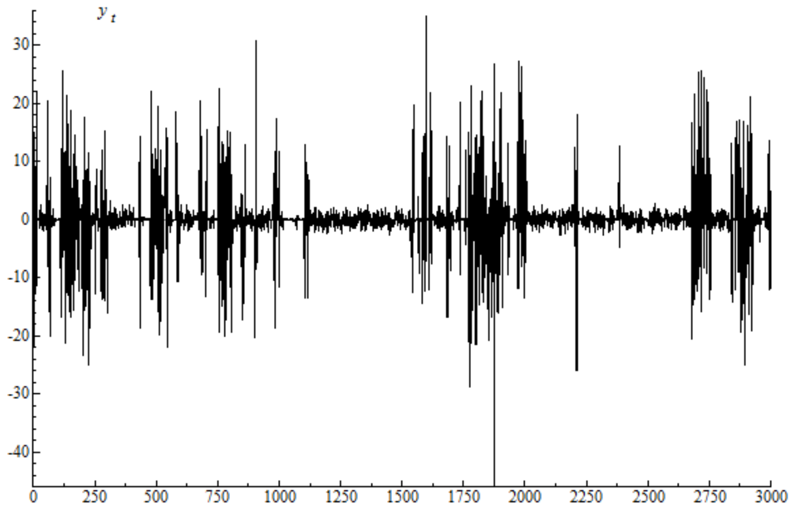

Figure B.1

Estimation with a 2-state Markov switching volatility model, gave the following output in the usual notation in terms of the transition matrix $P = (p_{ij})_{i,j=1,2}$ and smoothed standardized residuals $\hat{z}_t^*$:

$\hat{P}$, QMLE of P:	$\hat{p}_{11} = 0.98$ $\hat{p}_{21} = 0.044$
Normality test for $\hat{z}_t$:	p-value: 0.000
LM ARCH test in $\hat{z}_t$:	p-value: 0.000

What would you conclude on the basis of the output and the graph?

Question B.2: As an alternative to the 2-state Markov switching volatility model consider instead the 3-state switching model for log-returns y_t given by,

$$y_t = \sigma_{s_t} z_t$$

with z_t iidN$(0, 1)$ and $s_t = 1, 2$ or 3 such σ_{st} can take three values, σ_1, σ_2 and σ_3 ("low", "medium", "high" say) according to the value of the switching variable s_t.

Initially, we let the switching variable s_t be iid with $P(s_t = 1) = p_1$, $P(s_t = 2) = p_2$ and finally $P(s_t = 3) = p_3 = 1 - p_1 - p_2$.

Discuss if $\sigma_{s_t}^2$ and y_t are weakly mixing and stationary if the three probabilities p_1, p_2 and p_3 satisfy $0 < p_i < 1$.

Explain how you would write an algorithm in for example ox which can simulate the s_t series given fixed values of p_1 and p_2. You do not need to write the actual ox code, but should explain how you simulate s_t by drawing suitable iid series.

Question B.3: Given return data $Y := (y_t)_{t=1,2,3,...,T}$ we wish to estimate the parameters in $\theta = \{\sigma_1^2, \sigma_2^2, \sigma_3^2, p_1, p_2\}$.

Let $L(Y, S; \theta)$ be the likelihood function with both Y and $S := (s_t)_{t=1,2,...,T}$ observed. Find $L_{EM}(Y; \theta)$ where,

$$L_{EM}(Y; \theta) = E_{\tilde{\theta}}(\log L(Y, S; \theta) | Y).$$

Explain in particular how the so-called smoothed probabilities $(p_{it}^*)_{i=1,2,3}$ enter in $L_{EM}(Y; \theta)$.

Question B.4: Show that maximizing $L_{EM}(Y; \theta)$ one gets,

$$\hat{\sigma}_1^2 = \frac{\sum_{t=1}^T p_{1,t}^* y_t^2}{\sum_{t=1}^T p_{1,t}^*}.$$

Give an interpretation of $\hat{\sigma}_1^2$. Also explain how p_{1t}^* is computed.

Question B.5: It is found that a better description for the data at hand is given by letting s_t be Markov switching according to the transition matrix,

$$P = \begin{pmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \end{pmatrix}, \tag{4}$$

where $\sum_{i=1}^3 p_{1i} = \sum_{i=1}^3 p_{2i} = \sum_{i=1}^3 p_{3i} = 1$. It is empirically found that $H_0 : \sigma_2 = \sigma_3 = \sigma^*$ holds.

Interpret and comment on H_0.

Find under the assumption that H_0 holds, the two probabilities:

$$P(\sigma_t = \sigma^* | \sigma_{t-1} = \sigma_1) \quad \text{and} \quad P(\sigma_t = \sigma^* | \sigma_{t-2} = \sigma_1).$$